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THE ASYMPTOTIC FORM OF THE UNEVENLY HEATED FREE BOUNDARY OF A CAPILLARY FLUID AT LARGE MARANGONI NUMBERS*

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Formal asymptotic expansions of the solution of the stationary problem of the thermocapillary flow of fluid in an unbounded region, with the free boundary unevenly heated, are constructed for large values of the Marangoni number. A non-linear boundary layer is formed near the free surface, and selfmodelling solutions are found for this layer near the critical point. A slow flow outside the boundary layer satisfies the equations of an ideal fluid. An equation describing the free boundary is obtained. When the temperature gradient vanishes, this equation becomes the well-known equation of the equilibrium of the free boundary of a capillary fluid. Numerical computations are carried out to determine the form of the meniscus at the vertical solid wall, the free boundary of the fluid poured onto a horizontal surface for the plane and axisymmetric case, and the surface of a gas bubble adjacent to the wall in a heated fluid.

The non-linear equations of the stationary boundary Marangoni layer near the free boundary of a fluid unevenly heated because of the thermocapillary effect were formulated in /1/ and studied earlier /2-6/. Asymptotic expansions of the solution of the stationary problem of a low-viscosity fluid flow under the action of tangential stresses were constructed in /7/.

1. Consider the stationary problem of the flow of an incompressible fluid in an unbounded region D under the action of thermocapillary forces caused by uneven heating of the free surface Γ, for the system of Navier-Stokes equations, with vanishing viscosity $\nu \to 0$

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} + \mathbf{g},\ \operatorname{div}\mathbf{v} = 0 \tag{1.1}$$

$$p = 2\nu\rho\mathbf{n}\cdot\Pi\cdot\mathbf{n} + \sigma(\varkappa_1 + \varkappa_2) + p_*,\ \ 2\nu\rho\Pi\cdot\mathbf{n} - 2\nu\rho\,(\mathbf{n}\cdot\Pi\cdot\mathbf{n})\,\mathbf{n} = \nabla_1\sigma,\ (x, y, z) \in \Gamma;\ \mathbf{v}\cdot\mathbf{n}\,|_\Gamma = 0,\ \mathbf{v}\,|_L = 0 \tag{1.2}$$

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Here $\mathbf{v} = (v_x, v_y, v_z)$ is the velocity vector, $\mathbf{g} = -g\mathbf{e}_z$, $\mathbf{e}_z$ is the unit vector of the z axis, g is the acceleration due to gravity, $\mathbf{n}$ is the unit vector of the outer normal to the free boundary, Γ, Π is the deformation rate tensor, $\varkappa_1$ and $\varkappa_2$ are the principal curvatures of the surface Γ, p_* is the given pressure on Γ, $\nabla_1 = \nabla - (\mathbf{n}\cdot\nabla)\,\mathbf{n}$ is the gradient along Γ, σ is the surface tension, assumed to be a linear function of the temperature $\sigma = \sigma_0 + \sigma_T(T - T_*)$ where σ_0, σ_T, T_* are known constants, $\sigma_T < 0$ and the temperature is specified along Γ; L is the solid boundary. The velocity field vanishes at infinity.

When the viscosity is low, non-linear boundary layers form near the free boundary and the solid wall. In an unbounded region the flow, everywhere outside the boundary layer, is described approximately by Euler's equations. Below, we construct the formal asymptotic expansions of the solution of problem (1.1), (1.2) as $\nu \to 0$. The problem reduces to dimensionless form and a small parameter $\varepsilon = M^{-1/3}$ is introduced in which $M = |\sigma_T|\, d^2 A \rho^{-1} \nu^{-2}$ is the Marangoni number while d and A are the characteristic scales of the length and the temperature gradient. We note that when ε are small, we have small ν or large temperature gradients. The dimensionless pressure p' is defined by the relation $p = Pp' + p_0$ where $p_0 = -\rho g z$ is the hydrostatic pressure and $P = \sigma_0 d^{-1}$ is the scale of the pressure. The characteristic velocity $U = (\sigma_T^2 A^2 d \nu^{-1} \rho^{-2})^{1/3}$ within the boundary layer near the free surface is taken as the scale of the velocity. The asymptotic expansions of the solution of Problems (1.1), (1.2) are constructed in the form

$$\mathbf{v} \sim \mathbf{h}_0 + \varepsilon(\mathbf{h}_1 + \mathbf{v}_1 + \mathbf{w}_1) + \ldots, \ \zeta \sim \zeta_0 + \varepsilon\zeta_1 + \ldots, \ p' \sim \lambda q_0 + \lambda\varepsilon(p_1 + q_1 + r_1) + \ldots; \ \lambda = |\sigma_T|\, A d \sigma_0^{-1} \tag{1.3}$$

Here λ is the "capillary constant" /3/ and $z = \zeta(x, y)$ is the equation of the free boundary. Let D_Γ be the region of the boundary layer near the free surface, and D_L the region near the solid wall. Then $\mathbf{h}_k, q_k$ will be functions of the type representing the solutions of the problem of the boundary layer in the region D_Γ; $\mathbf{w}_1, r_1$ in the region D_L, and $\mathbf{v}_1, p_1$ will determine the solution of the problem outside the regions D_Γ, D_L. We note that the velocity scale, the orders of principal terms in the expansion (1.3), and the order of the thickness of the boundary layer in the region D_Γ are all found from the condition of equality of the orders of the viscous and inertial terms in the Navier-Stokes system, and in the boundary conditions (1.2) for tangential stresses. In this case the thickenss of the boundary layer in the region D_Γ is of the order of ε.

2. The boundary value problem for the principal terms of the asymptotic expansion (1.3) describing the flow in the boundary layer near the free surface, is obtained by applying to system (1.1), (1.2) the second iterative process using the method of Vishik-Lyusternik /8/. We introduce the local orthogonal ξ, φ, θ-coordinates near the surface Γ, using the formulas

$$x = X(\varphi, \theta) - \xi n_x, \quad y = Y(\varphi, \theta) - \xi n_y, \quad z = Z(\varphi, \theta) - \xi n_z$$

Here $x = X(\varphi, \theta)$, $y = Y(\varphi, \theta)$, $z = Z(\varphi, \theta)$ is the parametric equation of the surface Γ, ξ is the distance between the point (x, y, z) and the surface Γ; n_x, n_y, n_z are the components of the vector $\mathbf{n}$. The surfaces $\varphi = \text{const}$, $\theta = \text{const}$ form two families of orthogonal surfaces, which are chosen so that the lines of their intersection with Γ form the lines of principal curvatures. We assume that the segments of the normal to Γ do not intersect at sufficiently small ξ.

Let $h_{\varphi k}, h_{\theta k}, h_{\xi k}$ be the components of the vector $\mathbf{h}_k$ in local coordinates. We substitute (1.3) into (1.1) and (1.2), expand $\mathbf{v}_1, p_1$ in a Taylor series in powers of ξ, and introduce the stretching transformation $\xi = \varepsilon s$. Equating the coefficients of $\varepsilon^{-1}, \varepsilon^0$ to zero, we conclude that $h_{\xi 0} = 0$, and $h_{\varphi 0}, h_{\theta 0}$ satisfy the Prandtl boundary layer equations.

Let us write a boundary value problem for $h_{\varphi 0}, H_{\xi 1}$ for the axisymmetric and the plane case

$$\begin{gathered} g_\varphi^{-1} h_{\varphi 0} \frac{\partial h_{\varphi 0}}{\partial \varphi} + H_{\xi 1} \frac{\partial h_{\varphi 0}}{\partial s} = \frac{\partial^2 h_{\varphi 0}}{\partial s^2}, \quad \frac{\partial (h_{\varphi 0} g_\theta)}{\partial \varphi} + \frac{\partial}{\partial s}(g_\varphi g_\theta H_{\xi 1}) = 0 \\ \frac{\partial h_{\varphi 0}}{\partial s} = -g_\varphi^{-1} \frac{\partial \sigma}{\partial \varphi}, \quad H_{\xi 1} = 0 \ (s = 0); \quad h_{\varphi 0} = 0 \quad (s = \infty) \\ H_{\xi 1} = h_{\xi 1} + v_{\xi 1}|_\Gamma \end{gathered} \tag{2.1}$$

Here g_φ, g_θ are the Lamé coefficients of the surface Γ. We note that in the plane case $g_\theta = 1$, and in the axisymmetric case $g_\theta = r$, where r is the distance from the axis of symmetry. If φ is the arc length along Γ, then $g_\varphi = 1$. The vector function $\mathbf{h}_1$

satisfies a linear boundary value problem which is not quoted here.

We note that problem (2.1) for a plane boundary layer on a segment $\varphi \in [0, l]$ was studied for the given initial velocity profile $h_{\varphi 0} = f(s)$ $(\varphi = 0)$ in /6/, where the conditions of solvability were found.

Let us now give the selfsimilar solutions of Problem (2.1) for the axisymmetric case near the critical point. We note that the solutions were found in /2/ for the plane case, and in /5/ the Mangler transformations for the axisymmetric boundary layers near the free surface were obtained. Let the surface tension depend on the coordinate φ in accordance with the power law $\partial\sigma/\partial\varphi = a\varphi^n$ where φ is the arc length. Let us put in (2.1) $g_\varphi = 1, g_\theta = r$ and assume that $r = r_0\varphi$. We introduce the stream function ψ using the relations $h_{\varphi 0} = \partial\psi/\partial s$, $H_{\xi 1} = -r^{-1}\partial(r\psi)/\partial\varphi$, and write $\eta_1 = s\varphi^{(n-1)/3}$. Writing ψ in the form $\psi = \varphi^{(n+2)/3}F_1(\eta_1)$ we construct the following boundary value problem for $F_1(\eta_1)$ (the initial profile need not be given, since it is determined by the condition of selfsimilarity):

$$3F_1''' + (n+5)F_1F_1'' - (2n+1)F_1'^2 = 0 \tag{2.2}$$

$$F_1(0) = F_1'(\infty) = 0, \quad F_1''(0) = -a$$

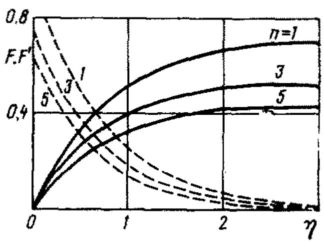

Fig.1

Integrating Eq.(2.2) on the semi-axis $(0, \infty)$ we find, that $a(n+2) > 0$. When $n = 4$, we have the exact exponential solution

$$F_1 = {}^1/_3(3a)^{1/3}[1 - \exp(-(3a)^{1/3}\eta_1)]$$

Problem (2.2) was integrated numerically for various values of n. Fig.1 shows the profiles of the functions $F(\eta)$ (the solid lines) and $F'(\eta)$ (the dashed lines) where $\eta = a^{1/3}\eta_1$ and $F = a^{-1/3}F_1$. The functions $F(\eta)$ increase monotonically and tend, as $\eta \to \infty$, to finite limits 0.7124; 0.5341 and 0.4386 for $n = 1; 3; 5$ respectively.

Next we shall determine the principal term of the asmyptotic expansion for the pressure in (1.3). Applying the second iterative process /8/ to system (1.1) projected onto the normal to the free surface, we obtain an equation for q_0, from which it follows that

$$q_0 = -\varkappa_1\int_s^\infty h_{\varphi 0}^2 ds - \varkappa_2\int_s^\infty h_{\theta 0}^2 ds \tag{2.3}$$

Let us find the value of q_0 on the free surface in the plane case. We put $\varkappa_2 = 0$ in (2.3), integrate the first equation of (2.1) with respect to s on the semi-axis $[0, \infty)$ integrate by parts and integrate the resulting expression with respect to φ. As a result we obtain

$$\int_0^\infty h_{\varphi 0}^2 ds = \sigma(\varphi) - \sigma(\varphi_0) + \gamma, \quad \gamma = \int_0^\infty f_0^2 ds \tag{2.4}$$

Here $f_0 = h_{\varphi 0}(s, \varphi_0)$ is the velocity profile in the boundary layer in the cross-section $\varphi = \varphi_0$. We note that it is convenient to choose φ_0, such that the value of f_0 is known. Writing $s = 0$ in (2.3) and taking into account relation (2.4), we obtain

$$q_0|_\Gamma = -\varkappa_1[2\sigma(\varphi) - \sigma(\varphi_0) + \gamma] \tag{2.5}$$

Similar reduction in the axisymmetric case leads to the relation

$$q_0|_\Gamma = -\varkappa_1 g_\theta^{-1}\left[\int_{\varphi_0}^{\varphi} g_\theta \frac{\partial\sigma}{\partial\varphi}d\varphi + \gamma g_\theta|_{\varphi=\varphi_0}\right] \tag{2.6}$$

Thus if the velocity profile in the boundary layer is known for some φ_0, the pressure at the free boundary can be determined to within infinitesimals of the order of $O(\varepsilon)$, without having to solve the problem for the boundary layer (2.1).

The functions v_1, p_1, ζ_0 determining the flow outside the region of the boundary layer and the asymptotic form of the boundary layer are obtained by applying the first iterative process /8/ to system (1.1), (1.2). For v_1, p_1 we obtain the Euler equations of an ideal fluid. Let us denote by Γ_0 the free boundary of an inviscid flow determined by the equation $z = \zeta_0$. We introduce near Γ_0 the local orthogonal coordinates $\xi_1, \varphi_1, \theta_1$, where ξ_1 is the distance from Γ_0. We represent the principal curvatures of the surface Γ in the form of series $\varkappa = \varkappa_{10} + \varepsilon\varkappa_{11} + \ldots$, $\varkappa_2 = \varkappa_{20} + \varepsilon\varkappa_{21} + \ldots$, where $\varkappa_{10}, \varkappa_{20}$ are the principal curatures of the surface Γ_0. Substituting the expansions (1.3) into the dynamic condition for the normal stress (1.2) and equating to zero the coefficient of ε^0, we obtain the equation of the boundary Γ_0. Taking (2.5) into account, we shall write this equation for the plane problem in dimensional form

$$\varkappa_{10}\,[2\sigma(\varphi) - \sigma(\varphi_0) + \gamma] = \rho g z + c \tag{2.7}$$

Let us write the equation for Γ_0 for the axisymmetric case

$$\sigma(\varkappa_{10} + \varkappa_{20}) + \varkappa_{10} g_\theta^{-1}\left[\int_{\varphi_0}^{\varphi} g_\theta \frac{\partial\sigma}{\partial\varphi}\,d\varphi + \gamma g_\theta|_{\varphi=\varphi_0}\right] = \rho g z + c \tag{2.8}$$

We note that when $\lambda = 0$ ($\sigma = \mathrm{const}$), Eqs.(2.7) and (2.8) will determine the free equilibrium surface of the capillary fluid in the gravity force field /9/ without heating. The inviscid flow outside the boundary layers is determined by solving the boundary value problem

$$(\mathbf{v}_1\cdot\nabla)\,\mathbf{v}_1 = -\nabla p_1, \quad \operatorname{div}\mathbf{v}_1 = 0$$
$$\mathbf{v}_1\cdot\mathbf{n}_0|_{\Gamma_0} = -H_{\xi 1}|_{\xi=\infty}; \quad \mathbf{v}_1\cdot\mathbf{n}_1|_L = 0, \quad \mathbf{v}_1|_\infty = 0$$

where $\mathbf{n}_0$ is the unit vector of the outer normal to Γ_0, $\mathbf{n}_1$ is the vector of the normal to the solid wall.

The vector function $\mathbf{w}_1$ determines the velocity field in the boundary layer near the solid boundary L, and compensates for the discrepancy arising when the vector $\mathbf{v}_1$ satisfies the adhesion condition on L. The boundary value problem for $\mathbf{w}_1, r_1$ is not given, since these functions contribute to the equation of the free surface in the higher-order approximations, beginning with the second.

3. Let us consider the case in which the equation of the free boundary is given in terms of quadratures. We shall assume that the flow of fluid is planar, and that there is no force of gravity ($g = 0$). Let φ be the arc length of the contour Γ_0, and let us write (2.7) in parametric form. We shall write the equation of the line Γ_0 in the form $x = x(\varphi)$, $z = z(\varphi)$ and denote by $\beta(\varphi)$ the angle of inclination of the line element Γ_0 obtained when φ increases. Then $x' = \cos\beta$, $z' = \sin\beta$. The equation of the boundary of Γ_0 will now take the form

$$[2\sigma(\varphi) - \sigma(\varphi_0) + \gamma]\,x'' = \pm c z'$$
$$[2\sigma(\varphi) - \sigma(\varphi_0) + \gamma]\,z'' = \pm c x', \quad c = \mathrm{const}$$

where the upper and lower sign is chosen for the fluid situated below the surface Γ_0 relative to the z axis, or above it. It can be shown that $\beta(\varphi)$ satisfies the relation

$$\frac{d\beta}{d\varphi} = \frac{\pm c}{2\sigma(\varphi) - \sigma(\varphi_0) + \gamma}$$

Now, having found $\beta(\varphi)$, we obtain the equation of the boundary

$$x = c_1 + \int_0^{\varphi}\cos\beta\,d\varphi, \quad z = c_2 + \int_0^{\varphi}\sin\beta\,d\varphi$$
$$\beta = c_3 \pm c\int_0^{\varphi}[2\sigma(\varphi) - \sigma(\varphi_0) + \gamma]^{-1}\,d\varphi$$

Here c_1, c_2 determine the Cartesian coordinate of the reference point N from which the arc length φ is measured, and the constant c_3 is the angle of inclination of the tangent to Γ_0 at the point N. The constant c is determined from the additional conditions in each particular case.

4. Let us determine the form of the free surface in the gravity field in the case when the fluid is in contact with a solid vertical wall $x = 0$ on one side $x \geqslant 0$. We shall assume that a negative temperature gradient $\partial T/\partial\varphi < 0$ is specified along Γ. We shall count the parameter φ (arc length of Γ) from the wall in the direction of increasing x.

We shall find the temperature distribution from the formula $T - T_* = AdG(\varphi)$, and will represent, in this case, the surface tension in the form $\sigma = \sigma_0(1 - \lambda G(\varphi))$, where $\lambda = |\sigma_T A| d\sigma_0^{-1}$ is the capillary constant. We shall assume everywhere below that the function $G(\varphi)$ is piecewise linear: $G(\varphi) = 1 - \varphi$ $(0 \leqslant \varphi \leqslant 1)$, $G = 0$ $(\varphi > 1)$, and here we have also assumed that $f_0(s) = 0$ in (2.7).

The equation of the boundary layer (2.1) has in the planar case for $0 \leqslant \varphi \leqslant 1$, a self-similar solution $h_{\varphi 0} = \varphi^{1/2} d\Phi(\eta)/d\eta$ where $\eta = s\varphi^{-1/2}$ and $\Phi(\eta)$ satisfies the boundary value problem (2.2) in which the coefficients $n + 5$ and $2n + 1$ should be replaced, respectively, by 2 and 1. The problem for $\Phi(\eta)$ was solved in /2/ by numerical methods. Choosing $\varphi_0 = 0$, we find that $f_0 = h_{\varphi 0}|_{\varphi=0} = 0$. We can now write the equation of the free boundary (2.7) in the following dimensionless form:

$$\xi_0''(1 + \xi_0'^2)^{-3/2}(1 + \lambda G_1(\varphi)) = \xi_0 B + c \tag{4.1}$$
$$B = \rho g d^2 \sigma_0^{-1}, \quad G_1 = G(0) - 2G(\varphi)$$

where B is the Bond number and a prime denotes a derivative with respect to x.

Let us formulate for Eq.(4.1) the boundary conditions $\xi_0'(0) = \operatorname{tg}\beta_0, \xi_0(\infty) = 0$, where β_0 is the angle between the tangent to Γ and the x axis at the point of contact $x = 0$.

Eq.(4.1) we integrated numerically for various values of λ and β_0. Fig.2 (the solid lines) shows the form of the free boundary for $B = 1$, $\beta_0 = -\pi/3$ and various values of λ. The height of the meniscus $h = \xi_0(0)$ decreases as λ increases and reaches a value of 0.414 at $\lambda = 0{,}99$. We note that $h = 2|\sin {}^1/_2\beta_0|$ when $\lambda = 0$ (when there is no temperature gradient). The equation of the free boundary at $\lambda = 0$ can be expressed in finite form in terms of elementary functions /9/.

In the special case when $|\beta_0| \ll 1, |\xi_0'| \ll 1$, Eq.(4.1) can be linearized and it has the following solution:

$$\xi_0 = (1 + 2\lambda x)^{1/2} B^{-1/2} [A_1 K(\lambda^{-1}\sqrt{B(1 + 2\lambda x)}) +$$
$$A_2 I_1(\lambda^{-1}\sqrt{B(1 + 2\lambda x)})] \quad (0 \leqslant x \leqslant 1)$$
$$\xi_0 = A_3 \exp(-x\sqrt{B(1 + 2\lambda)}) \quad (x > 1))$$

where $K_1(t), I_1(t)$ are modified Bessel functions. The constants A_1, A_2, A_3 can be expressed in terms of $\operatorname{tg}\beta_0$, and are not given here because of their complexity.

Let us determine the form of the free boundary of a fluid poured onto a solid horizontal wall and wetting its surface in the half-space $x \geqslant 0$. The wetting line is $x = 0, z = 0$. Let β_0 be the wetting angle /9/. The equation of the free boundary is obtained by integrating (4.1) under the conditions $\xi_0'(0) = \operatorname{tg}\beta_0$, $\xi_0'(\infty) = 0$.

In the course of numerical integration, Eq.(4.1) is written in parametric form using the arc length as parameter, and the constant c which appears in the expression for the layer thickness $H = -cB^{-1}$ is determined. Fig.3 shows the dependence of the layer thickness on the magnitude of the wetting angle $H(\beta_0)$ for $\lambda = 0; 0.333; 0.5$, with Bond numbers equal to 1: 0.666; 0.5 respectively (curves *1*, *2* and *3*). We note that when $\beta_0 = \text{const}$ and λ is fixed, the thickness H increases as B decreases, while when $B = \text{const}$, the thickness H decreases as λ increases (we note that $H = 1.17$ when $\lambda = 0.99$ and $B = 1$). The dashed lines in Fig.2 show the form of the free boundary at $H = 1.9$ and various vales of λ. When H is given and $B = \text{const}$, the wetting angle increases as the temperature gradient increases.

We note that (1.3) does not contain any functions of the boundary layer manifesting themselves near the line of contact between the free boundary and the solid wall. Here the asymptotic expansions are more complicated than (1.3). The equation of the boundary layer in this region is identical with the complete Navier-Stokes equations. The functions of the boundary layer contribute towards the equation of the free boundary layer only in the higher approximations, and are therefore not given here. An asymptotic study of the Navier-Stokes system near the line of contact is given in /10, 11/.

5. The asymptotic form of the free boundary is determined, in the axisymmetric case, with help of Eq.(2.8). Let us determine the form of the aperture in an infinite layer of fluid poured onto a horizontal plane. Let the temperature distribution $T - T_* = AdG(\varphi)$, be specified along the surface Γ where the function $G(\varphi)$ has been defined in the previous case and φ is the arc length of the surface Γ in the axial cross-section measured from the line of contact between Γ and the wall. Choosing as before $\varphi_0 = 0$, we conclude that $f_0(s) = 0$, and we write Eq.(2.8) in dimensionless form

$$\frac{(rz')'}{rr'}(1 - \lambda G(\varphi)) + \frac{r'z'' - z'r''}{r}\int_0^{\varphi} r\frac{\partial\sigma}{\partial\varphi}\,d\varphi = c + zB \tag{5.1}$$

$$r'^2 + z'^2 = 1$$

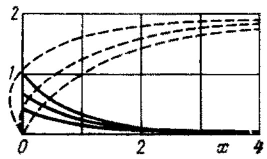

Fig.2

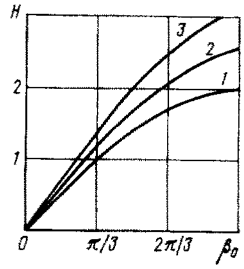

Fig.3

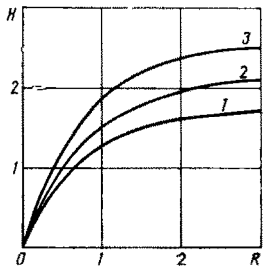

Fig.4

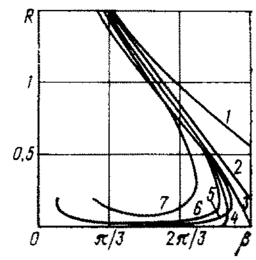

Fig.5

where $r(\varphi)$, $z(\varphi)$ are the cylindrical coordinates of the surface Γ. We write the boundary conditions in the form $r(0) = R$, $z(0) = 0$, $r'(0) = \cos\beta_0$, $z'(0) = \sin\beta_0$, where β_0 is the wetting angle and R is the radius of the wetting line of Γ and the wall. The constant c in (5.1) is not known, and is determined with help of the additional condition $z'(\infty) = 0$. The height of the fluid layer is determined from the formula $H = -cB^{-1}$, and system (5.1) was integrated numerically using the Runge-Kutta method.

Fig.4 shows the dependence of the layer height on the radius of the wetting line at $\beta_0 = 150°$. The Curves *1*, *2* and *3* were constructed with the values of $\lambda = 0; 0.333; 0.5$, for Bond numbers equal to 1; 0.666; 0.5 respectively. When λ and B are fixed, the height H increases as R increases and reaches its limit value at $R = \infty$. When B and H are fixed, the angle β_0 increases as λ increases.

6. Let us determine the form of a gas bubble adhering to the horizontal solid wall in a non-uniformly heated fluid within the gravity field. System (1.1) is solved together with the equation of heat conduction $\Pr \mathbf{v}\cdot\nabla T = \varepsilon^2 \Delta T$, where Pr is the Prandtl number. When $\varepsilon \to 0$, the temperature T can be expanded in a series of the form (1.3) with coefficients T_k, θ_k, where T_k is the outer solution and θ_k satisfies the equation of the boundary temperature layer.

We shall assume that a constant temperature gradient $\nabla T_0 = A\mathbf{e}_z$ is specified outside the boundary layer. We find that at small Prandtl numbers $\Pr \ll 1$, $\theta_0 = 0$. Then the dimensionless surface tension will have the form $\sigma = 1 + \lambda z$ with the accuracy of up to $O(\varepsilon)$. The form of the free boundary is determined by Eq.(2.8), where we assume that $\varphi_0 = 0$, i.e. we choose the cross-section in the boundary layer on the axis of symmetry. It can be shown that $f_0 = 0$ in (2.8).

We note that when $\varphi \to 0$, (2.8) yields

$$z = {}^1/_4 c\varphi^2 + \ldots, \quad r = r_0\varphi + \ldots, \quad \partial\sigma/\partial\varphi = {}^1/_2 c\varphi + \ldots$$

(r is the distance along the axis of symmetry and φ is the arc length of Γ). The solution of the equation of the boundary layer can be expanded near the critical point in a series in powers of φ, with the coefficients depending on s, $h_{\varphi 0} = \varphi F_1'(s) + \ldots$, where the function $F_1(s)$ satisfies Eq.(2.1) and $n = 1$. This implies that $f_0 = 0$ in (2.8).

Thus the form of free boundary layer $z = \xi_0(r)$ satisfies the equation

$$\frac{\xi_0''}{(1+\xi_0'^2)^{3/2}}\left(1 + 2\lambda\xi_0 - \frac{\lambda}{r}\int_0^r \xi_0 dr\right) + \frac{(1+\lambda\xi_0)\xi_0'}{r\sqrt{1+\xi_0'^2}} = B\xi_0 + c \tag{6.1}$$

When numerical integration is carried out, Eq.(6.1) is written in parametric form with the arc length used as the parameter. The angle β, where $\pi-\beta$ is the wetting angle, the constant λ, the dimensionless volume of the bubble $V=1$, and the Bond number were all specified, while the constant c and the radius R of the wetting line were determined.

When $r \to 0$, the solution was expanded in a series in powers of φ /9/ and it coalesced with the numerical solution.

Fig.5 shows the dependence of the radius R of the wetting line on the angle β for $B=0$; $B=0.1$ and $B=1$. The curves *1* $(B=1)$ and *2* $(B=0.1)$ correspond to the absence of a temperature gradient $\lambda=0$. When λ increases while β and B are fixed, the radius R decreases and vanishes for some values of $\lambda=\lambda_*$ and $c=c_*$. For example, $\lambda_*=0.022$; $c_*=2.182$ for $B=0.1$; $\beta=\pi$ and $\lambda_*=0.234$; $c_*=2.063$ for $B=1$; $\beta=\pi$. Curves *3* and *4* correspond to $\lambda=0.022$; $B=0.1$ and $\lambda=0.3$; $B=1$. The bubble becomes detached from the wall when $\lambda=\lambda_*, c=c_*$. For $\lambda>\lambda_*$ the bubble can also become detached from the wall, and will be in a state of equilibrium under the action of gravity and thermocapillary forces, while the free surface will have a cusp. Note that the radius R may become zero when $\lambda_1 \leqslant \lambda_* \leqslant \lambda_2$, for example $\lambda_1=0.022$ and $\lambda_2=0.047$ when $B=0.1$. Curve *5* corresponds to $\lambda=0.047$ $(B=0.1)$. When $\lambda>\lambda_1$, the integral curves of the equation may intersect the z axis in the r, z-plane for certain values of c. This does not happen when there is no temperature gradient /9/. Part of the curves has a form typical for the integral curves of the free equilibrium boundary $(\lambda=0)$ for positive overloads $(g>0)$, and another part for negative overloads /9/.

When there are no gravitational forces $(B=0)$ $R\neq 0$ for all β, and $\lambda>0$. The bubble adheres to the wall under the action of the thermocapillary forces. Curves *6* and *7* in Fig.5 correspond to the value $\lambda=0.01$ and $\lambda=0.1$. When λ is fixed, there exists a maximum value of the angle of contact β_m, for example $\beta_m=163°$ when $\lambda=0.01$. Computations have shown that when β is given, the free surface is stretched along the z axis as λ and c increase.

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